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Some observations on the dynamics of the Aharonov–Bohm effect

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Abstract. It is often said that the Aharonov-Bohm (AB) effect is an effect that is due to the vector potential. This is considered remarkable since a quantity that is not uniquely determined has a measurable effect. We show that, in the symmetric Coulomb gauge, $(e/c)\boldsymbol{A}(\boldsymbol{r})$ is simply the momentum of the electromagnetic field. The effect may equally well be considered an effect of the electromagnetic momentum. Thus, it is possible to discuss the AB effect strictly in terms of observables. The idea that the effect is due to the vector potential is correct, but sheds little light on the physics of the effect.

1. Introduction

In a well known work published in the early days of quantum mechanics, Dirac [1] demonstrated that in a region free of magnetic fields, the solution of the Schrödinger equation may be written

$$\Psi(\boldsymbol{r}, t) = \Psi^0(\boldsymbol{r}, t) \exp\left(\frac{\mathrm{i}e}{\hbar c}\int^{\boldsymbol{r}} \boldsymbol{A}(\boldsymbol{r}') \cdot \mathrm{d}\boldsymbol{r}'\right) \tag{1}$$

where $\Psi^0(\boldsymbol{r}, t)$ satisfies the Schrödinger equation with the same scalar potential, but in which the vector potential has been set equal to zero. The phase factor of equation (1) is the basis of all derivations of the Aharonov-Bohm (AB) effect [2], that are not explicit scattering calculations. The importance of this phase factor has been emphasised by Wu and Yang [3, 4] and later by one of the present authors [5-7]. It is generally agreed that this phase factor plays an essential role in all AB-type problems.

2. Classical forces

In order to understand fully the forces involved in the AB effect, it is essential (reader, please bear with us!) to consider the effect of the electron's own electric and magnetic fields. The viewpoint that the interaction energy of the flux column with the electron is purely magnetic leads to an interaction energy

$$\Delta E = \frac{1}{4\pi}\int \boldsymbol{B}_{\mathrm{el}}(\boldsymbol{r}' - \boldsymbol{r}) \cdot \boldsymbol{B}_{\mathrm{ext}}(\boldsymbol{r}')\, \mathrm{d}^3 r' \tag{2}$$

with

$$\boldsymbol{B}_{\mathrm{el}} = \frac{1}{c}\, \boldsymbol{v} \times \boldsymbol{E}(\boldsymbol{r}' - \boldsymbol{r}) \tag{3}$$

for an electron at point $\boldsymbol{r}$ moving with velocity $\boldsymbol{v}$. It is useful to consider the quantity

$$\begin{aligned}\frac{1}{4\pi}\nabla\int \boldsymbol{B}_{\rm el}(\boldsymbol{r}'-\boldsymbol{r})\cdot\boldsymbol{B}_{\rm ext}(\boldsymbol{r}')\,{\rm d}^3r'\\ &=\frac{1}{4\pi c}\nabla\int \boldsymbol{v}\times\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\cdot\boldsymbol{B}_{\rm ext}(\boldsymbol{r}')\,{\rm d}^3r'\\ &=\frac{1}{4\pi c}\nabla\int \boldsymbol{v}\cdot\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\times\boldsymbol{B}_{\rm ext}(\boldsymbol{r}')\,{\rm d}^3r'\\ &=\nabla(\boldsymbol{v}\cdot\boldsymbol{P}_{\rm field})=(\boldsymbol{v}\cdot\nabla)\boldsymbol{P}_{\rm field}+\boldsymbol{v}\times(\nabla\times\boldsymbol{P}_{\rm field})\\ &=\frac{{\rm d}P_{\rm field}}{{\rm d}t}+\boldsymbol{v}\times(\nabla\times\boldsymbol{P}_{\rm field}).\end{aligned} \tag{4}$$

Here we have made use of the expression for the electromagnetic momentum, which depends upon the electric field of the electron and the magnetic field of the solenoid.

The reader may have guessed that the second term on the right-hand side of equation (4) is just $(e/c)\boldsymbol{v}\times\boldsymbol{B}_{\rm ext}$, the Lorentz force. This can be seen as follows:

$$\begin{aligned}\frac{1}{4\pi c}\nabla\times\int \boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\times\boldsymbol{B}_{\rm ext}(\boldsymbol{r}')\,{\rm d}^3r'\\ &=\frac{1}{4\pi c}\left(-\int \boldsymbol{B}_{\rm ext}(\boldsymbol{r}')[\nabla\cdot\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})]\,{\rm d}^3r'+\int[\boldsymbol{B}_{\rm ext}(\boldsymbol{r}')\cdot\nabla]\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\,{\rm d}^3r'\right).\end{aligned} \tag{5}$$

Making use of the fact that $\nabla\cdot\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})=-\nabla'\cdot\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})$ yields

$$\nabla\times\boldsymbol{P}_{\rm field}=\frac{1}{4\pi c}\left(\int \boldsymbol{B}(\boldsymbol{r}')4\pi e\delta(\boldsymbol{r}'-\boldsymbol{r})\,{\rm d}^3r'+\int[\boldsymbol{B}(\boldsymbol{r}')\cdot\nabla]\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\,{\rm d}^3r'\right). \tag{6}$$

The first term of equation (6) is $(e/c)\,\boldsymbol{B}(\boldsymbol{r})$. The second term can be shown to vanish. Let $\boldsymbol{\varepsilon}$ be an arbitrary constant vector:

$$\begin{aligned}\boldsymbol{\varepsilon}\cdot\int[\boldsymbol{B}(\boldsymbol{r}')\cdot\nabla')\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\,{\rm d}^3r'\\ &=\int[\boldsymbol{B}(\boldsymbol{r}')\cdot\nabla'][\boldsymbol{\varepsilon}\cdot\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})]\,{\rm d}^3r'\\ &=\int\nabla'\cdot[\boldsymbol{\varepsilon}\cdot\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})]\boldsymbol{B}(\boldsymbol{r}')\,{\rm d}^3r'-\int\boldsymbol{\varepsilon}\cdot\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\nabla'\cdot\boldsymbol{B}(\boldsymbol{r}')\,{\rm d}^3r'.\end{aligned} \tag{7}$$

The first integral can be converted into a surface integral that vanishes at infinity. The second integral vanishes since $\nabla\cdot\boldsymbol{B}=0$. Thus, we have the result

$$\frac{1}{4\pi}\nabla\int \boldsymbol{B}_{\rm el}(\boldsymbol{r}'-\boldsymbol{r})\cdot\boldsymbol{B}_{\rm ext}(\boldsymbol{r}')\,{\rm d}^3r'=\frac{{\rm d}}{{\rm d}t}\boldsymbol{P}_{\rm field}+\left(\frac{e}{c}\right)\boldsymbol{v}\times\boldsymbol{B}. \tag{8}$$

Equation (8) gives a new insight into the dynamics of the AB problem. The Lorentz force is simply the time rate of change of the kinetic momentum, $\boldsymbol{\pi}$. The left-hand side of equation (8) reads $\boldsymbol{F}=\nabla(\Delta E)$, and not $\boldsymbol{F}=-\nabla(\Delta E)$, where ΔE is given by equation (2). We assume that the currents in the solenoid windings as well as the probability current density of the electron are kept constant. (The physical consequences of this assumption are discussed fully in section 5.) It is well known [8] that the magnetic force between current carrying conductors is given by $\boldsymbol{F}=\nabla(\Delta E)$ when the currents are kept constant. Our force law now reads

$$\frac{1}{4\pi}\nabla\int \boldsymbol{B}_{\mathrm{el}}(\boldsymbol{r}'-\boldsymbol{r})\cdot\boldsymbol{B}_{\mathrm{ext}}(\boldsymbol{r}')\,\mathrm{d}^3r'=\frac{\mathrm{d}\boldsymbol{P}_{\mathrm{total}}}{\mathrm{d}t} \tag{9}$$

with $\boldsymbol{P}_{\mathrm{total}}=\boldsymbol{\pi}+\boldsymbol{P}_{\mathrm{field}}$. It is now clear that the usual statement that there is no force in the AB problem requires clarification. There is no *mechanical* force (i.e. no force on the inertial mass of the electron). However, the rate of change of the total momentum is not zero, since the electric and magnetic fields of the particle penetrate the solenoid. (In cases in which the interior of the solenoid is shielded, the electric and magnetic fields of the electron interact with currents in the shielding material.)

Since $\nabla\times\boldsymbol{P}_{\mathrm{field}}=(e/c)\nabla\times\boldsymbol{A}$, it is clear that there exists a gauge in which $(e/c)\boldsymbol{A}=\boldsymbol{P}_{\mathrm{field}}$. This is the 'natural' gauge in which to solve problems. We see that, because of this relation, the AB effect can be traced to the physical quantity that can be considered responsible for the effect. The gauge invariance of the theory then gives the appearance that a non-physical quantity (the vector potential) is responsible for the effect.

The result that the 'natural' gauge, in which $(e/c)\boldsymbol{A}(\boldsymbol{r})$ is equal to $\boldsymbol{P}_{\mathrm{field}}$ is the symmetric Coulomb gauge will probably not surprise the reader,

$$\begin{aligned}\boldsymbol{P}_{\mathrm{field}}(\boldsymbol{r})&=\frac{1}{4\pi c}\int \boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\times[\nabla'\times\boldsymbol{A}(\boldsymbol{r}')]\,\mathrm{d}^3r'\\&=\frac{1}{4\pi c}\int \nabla'[\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}')]\,\mathrm{d}^3r'-\frac{1}{4\pi c}\int[\boldsymbol{A}(\boldsymbol{r}')\cdot\nabla']\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\,\mathrm{d}^3r'\\&\quad-\frac{1}{4\pi c}\int[\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\cdot\nabla']\boldsymbol{A}(\boldsymbol{r}')\,\mathrm{d}^3r'-\frac{1}{4\pi c}\int \boldsymbol{A}(\boldsymbol{r}')\times[\nabla\times\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})]\,\mathrm{d}^3r'.\end{aligned} \tag{10}$$

The fourth term of equation (10) vanishes, since $\nabla\times\boldsymbol{E}=0$ for the Coulomb field of the electron. The first integral vanishes by a corollary of Gauss's theorem. The second integral can be shown to vanish in the Coulomb gauge. (The present derivation assumes that the external magnetic field has a finite source, so that the fields vanish at infinity.) Again, taking $\boldsymbol{\varepsilon}$ to be an arbitrary constant vector, one finds

$$\begin{aligned}&\boldsymbol{\varepsilon}\cdot\int[\boldsymbol{A}(\boldsymbol{r}')\cdot\nabla']\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\,\mathrm{d}^3r'\\&\qquad=\int\nabla'\cdot[\boldsymbol{A}(\boldsymbol{r}')\boldsymbol{\varepsilon}\cdot\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})]\,\mathrm{d}^3r'-\int[\boldsymbol{\varepsilon}\cdot\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})]\nabla'\cdot\boldsymbol{A}(\boldsymbol{r}')\,\mathrm{d}^3r'.\end{aligned} \tag{11}$$

Again, the first term of equation (11) can be converted into a vanishing surface integral. The second term vanishes if one takes $\nabla'\cdot\boldsymbol{A}=0$, the Coulomb gauge. The third integral of equation (10) can be shown to be

$$\frac{1}{4\pi c}\int \boldsymbol{A}(\boldsymbol{r}')\nabla'\cdot\boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\,\mathrm{d}^3r'=\frac{1}{c}\int\boldsymbol{A}(\boldsymbol{r}')e\delta(\boldsymbol{r}'-\boldsymbol{r})\,\mathrm{d}^3r'=\frac{e}{c}\boldsymbol{A}(\boldsymbol{r}). \tag{12}$$

This shows that, in a realistic problem involving sources of finite extent in space, the electromagnetic momentum is given by $(e/c)\boldsymbol{A}(\boldsymbol{r})$, where $\boldsymbol{A}(\boldsymbol{r})$ is the (uniquely defined) vector potential in the Coulomb gauge.

3. The infinite solenoid

When sources of the external flux are not confined to a finite region, the Coulomb gauge condition may not be unique. The vector potential satisfying $\boldsymbol{P}_{\text{field}}(\boldsymbol{r}) = (e/c)\boldsymbol{A}(\boldsymbol{r})$ for the ideal infinite solenoid is the symmetric Coulomb gauge. The proof follows.

In cylindrical polar coordinates, the symmetric Coulomb gauge is

$$\boldsymbol{A}(\boldsymbol{r}) = A(\rho)\hat{\theta} \quad \text{with } A(\rho) = \begin{cases} \frac{1}{2}B\rho & \rho \leqslant R \\ \Phi/2\pi\rho & \rho \geqslant R \end{cases}$$
$$\Phi = B\pi R^2. \tag{13}$$

Without loss of generality, we may consider the electron to be located a distance ρ from the origin along the x-axis. The cases $\rho \leqslant R$ and $\rho \geqslant R$ may be considered as special cases of the same computation:

$$\boldsymbol{P}_{\text{field}}(\boldsymbol{r}) = \frac{1}{4\pi c}\int \boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r}) \times \boldsymbol{B}(\boldsymbol{r}')\,\mathrm{d}^3r'$$
$$= -\frac{\boldsymbol{B}}{4\pi c} \times \int_V \boldsymbol{E}(\boldsymbol{r}'-\boldsymbol{r})\,\mathrm{d}^3r'. \tag{14}$$

In the last integral, V is the volume of the interior of the solenoid. By symmetry, $\int E_y\,\mathrm{d}^3r'$ and $\int E_z\,\mathrm{d}^3r'$ vanish. Therefore,

$$\boldsymbol{P}_{\text{field}}(\boldsymbol{r}') = -\hat{j}\frac{B}{4\pi c}\int_V E_x(\boldsymbol{r}'-\boldsymbol{r})\,\mathrm{d}^3r' = \hat{j}\frac{B}{4\pi c}\int_V \nabla'_x V(\boldsymbol{r}'-\boldsymbol{r})\,\mathrm{d}^3r'$$
$$= \hat{j}\frac{B}{4\pi c}\int_{\text{surface}} V(\boldsymbol{r}'-\boldsymbol{r})n'_x\,\mathrm{d}a'. \tag{15}$$

In equation (15), $\hat{j}$ is the unit vector in the y-direction and $n_x = \cos\theta'$. The Coulomb potential of the electron is given by

$$V(\boldsymbol{r}'-\boldsymbol{r}) = \frac{e}{|\boldsymbol{r}'-\boldsymbol{r}|} = \sum_{m=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}k\,\mathrm{e}^{-\mathrm{i}m\theta'}J_m(k\rho)J_m(kR)\,\mathrm{e}^{-k|z'|} \tag{16}$$

$$\boldsymbol{P}_{\text{field}}(\boldsymbol{r}) = \hat{j}\frac{Be}{4\pi c}\sum_{m=-\infty}^{\infty}\int_0^{\infty}\mathrm{d}k\int_{-\infty}^{\infty}\int_0^{2\pi}\cos\theta'\,\mathrm{e}^{-\mathrm{i}m\theta'}J_m(k\rho)J_m(kR)\,\mathrm{e}^{-k|z'|}R\,\mathrm{d}\theta'\,\mathrm{d}z' \tag{17}$$

Writing $\cos\theta' = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta'} + \mathrm{e}^{-\mathrm{i}\theta'})$, one obtains

$$\boldsymbol{P}_{\text{field}}(r') = \hat{j}\frac{BeR}{2c}\int_0^{\infty}\mathrm{d}k\int_{-\infty}^{\infty}J_1(k\rho)J_1(kR)\,\mathrm{e}^{-k|z'|}\,\mathrm{d}z' \tag{18}$$

where $J_{-1}(x) = -J_1(x)$ has been used. The z' integration yeilds

$$\boldsymbol{P}_{\text{field}} = \hat{j}\frac{BeR}{c}\int_0^{\infty}\frac{\mathrm{d}k}{k}J_1(k\rho)J_1(kR). \tag{19}$$

The integral of equation (19) yields the final result

$$\boldsymbol{P}_{\text{field}}(\boldsymbol{r}) = \hat{j}\frac{e}{c}\begin{cases} \frac{1}{2}B\rho & \text{for } \rho \leqslant R \\ \Phi/2\pi\rho & \text{for } \rho \geqslant R. \end{cases} \tag{20}$$

Thus, for the ideal infinite solenoid, $\boldsymbol{P}_{\text{field}}(\boldsymbol{r}) = (e/c)\boldsymbol{A}(\boldsymbol{r})$ when the vector potential is in the symmetric Coulomb gauge.

After this work was completed and submitted for publication, it was brought to our attention that a proof that $\boldsymbol{P}_{\text{field}}(\boldsymbol{r}) = (e/c)\boldsymbol{A}(\boldsymbol{r})$ for the infinite solenoid in the Coulomb gauge was given by Boyer [9] almost twenty years ago. It is truly unfortunate that the significance of Boyer's result was never fully appreciated.

4. The Dirac phase factor

We are now in a position to examine the relation of the interaction energy of equation (2) and the forces of equation (8) to the Dirac phase factor of equation (1). It is essential to remember that in quantum theory the wave properties of the electron are determined by the *canonical* momentum. In the symmetric Coulomb gauge, this is just the total momentum (kinetic momentum plus field momentum). Here we have consistently neglected terms involving $\boldsymbol{E}_{\text{el}} \times \boldsymbol{H}_{\text{el}}$. This momentum is related to an electromagnetic self-energy.

We begin by noting that the expectation value of the interaction energy in the quantum wave equations is

$$\Delta\tilde{E} = -\frac{e}{c}\int \boldsymbol{j}\cdot \boldsymbol{A}_{\text{ext}}(\boldsymbol{r})\,\mathrm{d}^3r. \tag{21}$$

Equation (21) is exact in Dirac theory and correct to first order in e in non-relativistic Schrödinger theory.

The current density is related to the proper field of the particle by

$$\nabla\times \boldsymbol{B}_{\text{el}} = \frac{4\pi e}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial E_{\text{el}}}{\partial t}. \tag{22}$$

The energy shift of equation (21) now becomes

$$\Delta\tilde{E} = -\frac{1}{4\pi}\int \nabla\times \boldsymbol{B}_{\text{el}}\cdot \boldsymbol{A}_{\text{ext}}(\boldsymbol{r})\,\mathrm{d}^3r + \frac{1}{4\pi c}\int \frac{\partial \boldsymbol{E}_{\text{el}}}{\partial t}\cdot \boldsymbol{A}_{\text{ext}}(\boldsymbol{r})\,\mathrm{d}^3r. \tag{23}$$

The second integral of equation (23) may be written

$$\frac{1}{4\pi c}\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{E}_{\text{el}}(\boldsymbol{r})\cdot \boldsymbol{A}_{\text{ext}}(\boldsymbol{r})\,\mathrm{d}^3r. \tag{24}$$

Next, we approximate the electric field of the particle by the instantaneous Coulomb field, that is retardation effects are neglected. This approximation has also been used by Peshkin [10]. In this approximation, we have a scalar product of a longitudinal electric field with a transverse (in the Coulomb gauge) vector potential integrated over all space. Such an integral vanishes.

The first term of equation (23) may be integrated by parts to yield

$$\Delta\tilde{E} = -\frac{1}{4\pi}\int \boldsymbol{B}_{\text{el}}\cdot \boldsymbol{B}_{\text{ext}}\,\mathrm{d}^3r \tag{25}$$

which is the negative of equation (2). The force is then given by $\boldsymbol{F} = -\nabla(\Delta\tilde{E})$, where $\boldsymbol{F}$ is the force of equation (9). Standard theory yields the *mechanical* force

$$\boldsymbol{F} = \frac{\mathrm{i}}{\hbar}[H, \boldsymbol{v}] = \frac{e}{2c}(\boldsymbol{v}\times \boldsymbol{B}_{\text{ext}} - \boldsymbol{B}_{\text{ext}}\times \boldsymbol{v}) \tag{26}$$

with $m\cdot \boldsymbol{v} = \boldsymbol{\pi} = (\hbar/\mathrm{i})\nabla - (e/c)\boldsymbol{A}(\boldsymbol{r})$.

The force of equation (26) vanishes identically on the space of Dirac wavefunctions [5]. A proper choice of wavefunctions is necessary to insure that the gradient of the magnetic field energy is equal to the time rate of change of electromagnetic momentum in the AB problem. The quantum version of equation (8) is not an operator identity. Its validity depends upon the electron probability current being related to the proper fields of the electron through Maxwell's equations.

The importance of the energy shift of equation (25) was first emphasised by Freeman [11] and by Coulson and Freeman [12]. These authors pointed out that, in a semi-classical argument, one can equally well treat the Aharonov-Bohm effect by replacing the Dirac phase factor by the phase factor

$$e^{-i/\hbar \int^t \Delta\tilde{E}\,dt}$$

with $\Delta\tilde{E}$ given by equation (25). The Freeman-Coulson (FC) phase, because of the equivalence of equations (21) and (25), is given by

$$\phi_{FC}=\frac{e}{\hbar c}\int^t \int \boldsymbol{j}(\boldsymbol{r}', t)\cdot \boldsymbol{A}_{ext}(\boldsymbol{r}')\,d^3r'\,dt. \tag{27}$$

In a semiclassical approximation in which $\boldsymbol{A}_{ext}(\boldsymbol{r})$ changes little in the region in which $\psi(r, t)$ has support, one may take $\boldsymbol{A}_{ext}(\boldsymbol{r})$ outside the integral, evaluating it at the centre of gravity of the wavepacket. Thus,

$$\begin{aligned}\phi_{FC}&=\frac{e}{\hbar c}\int^t \boldsymbol{A}_{ext}(\boldsymbol{r}, t)\cdot \int \boldsymbol{j}(\boldsymbol{r}', t)\,d^3r'\,dt=\frac{e}{\hbar c}\int^t \boldsymbol{v}\cdot \boldsymbol{A}_{ext}(\boldsymbol{r}, t)\,dt\\ &=\frac{e}{\hbar c}\int^r \boldsymbol{A}_{ext}(\boldsymbol{r}')\cdot d\boldsymbol{r}'.\end{aligned} \tag{28}$$

This last expression is $\phi_{Dirac}(\boldsymbol{r})$. The argument given here is semiclassical. It could be made rigorous by adopting a quantised field formulation in which $\boldsymbol{j}(\boldsymbol{r}, t)$ is an operator. The FC phase factor then becomes a sum of time-ordered products.

5. The ideal AB experiment

In most theoretical treatments of the AB effect, the flux along the z-axis is assumed (not including the flux due to the electron!) to be absolutely constant. Let us consider the consequences of this assumption. Assume an ideal solenoid of n turns per unit length. In order to maintain the solenoid flux constant, the current must be well regulated. The passing electron induces a voltage into the windings in dz given by

$$d\varepsilon = -\frac{n}{c}dz\frac{d}{dt}\int B_z(\boldsymbol{r})\,da \tag{29}$$

and the total voltage induced in the solenoid is

$$\varepsilon = -\frac{n}{c}\frac{d}{dt}\int_V B_z\,d^3r \tag{30}$$

where V is the volume of the solenoid. In order to maintain the current (and thus, the solenoid flux) fixed, the current source must produce an additional voltage given by the negative of equation (30). The source must therefore produce an additional power given by

$$\Delta P_{\text{inst}} = \frac{nI}{c}\frac{\mathrm{d}}{\mathrm{d}t}\int_V B_z\,\mathrm{d}^3r = \frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_V B_z B_0\,\mathrm{d}^3r. \tag{31}$$

Thus the additional field energy is supplied (of course!) by the current source of the solenoid. In the ideal AB experiment, *at a given instant*, this power is positive or negative depending on whether the electron passes to the right or to the left of the solenoid. Thus in the *ideal* AB effect, the external source itself becomes a quantum system, at any instant having a 50% chance of producing above average power and a 50% chance of producing less than average power. This macroscopic system becomes reminiscent of Schrödinger's cat, which had a 50% chance of being alive and a 50% chance of being dead [13].

An interesting mechanical model of the source of the magnetic field has been given by Peshkin *et al* [14] whose approach complements the one given here. However, we caution the reader that we do not share all of their views.

6. The physical AB effect

The conditions of the previous section are clearly not realisable in the laboratory. In particular, there are many magnetic fields acting on the solenoid that induce a larger voltage than the field of the passing electron. Among these are the fluctuation in the flux due to natural current fluctuations, the fields of other passing charged particles, and the field due to the electron's own magnetic moment. It is therefore clear that, in practice, the current source may be considered a classical source.

The fluctuations in the flux are only corrections to the flux. However, there appear to be so many that it is reasonable to wonder how the fringe pattern can be so stable. The stability of the fringes can be understood as follows. The wave properties in quantum theory depend upon the canonical momentum (in the symmetric Coulomb gauge, on the total momentum, which is the sum of $\boldsymbol{\pi}$ and $\boldsymbol{P}_{\text{field}}$). Let $\delta\Phi$ be the fluctuation in the flux that occurs in a time δt due to all causes. This fluctuation in the flux induces an electric field on an electron given by

$$E_\theta = -\frac{1}{2\pi rc}\frac{\delta\Phi}{\delta t}$$

so that

$$\delta\pi_\theta = eE_\theta\delta t = -\frac{e}{2\pi rc}\,\delta\Phi. \tag{32}$$

The change in P_{field} is (in the symmetric Coulomb gauge)

$$\delta P_{\theta_{\text{field}}} = \frac{e}{c}\,\delta A_\theta = \frac{e}{2\pi rc}\,\delta\Phi. \tag{33}$$

hence,

$$\delta P_{\text{total}} = \delta\pi + \delta P_{\text{field}} = 0. \tag{34}$$

The total momentum (and hence the local wavelength) is unaffected by variations in flux. Were it not for equation (34), observation of the AB effect might be impossible.

7. Summary

The principal results of this work are:

(i) In the symmetric Coulomb gauge, $(e/c)\boldsymbol{A}(\boldsymbol{r})$ is the electromagnetic momentum due to the Coulomb field of the electron in the external flux of the solenoid.

(ii) Gauge invariance causes effects of the electromagnetic momentum to appear as effects due to the vector potential.

(iii) The wave length of the electron wave is related to the canonical momentum (in the symmetric Coulomb gauge, the total momentum) of the electron. Thus, it is possible for interference effects depending on a shift in electromagnetic momentum to occur, even when the kinetic momentum is conserved.

(iv) Newton's second law states that, for the AB problem, $(e/c)\boldsymbol{v}\times\boldsymbol{B}=\dot{\boldsymbol{\pi}}$. There is no contradiction whatever between the AB effect and Newton's second law. This conclusion is further borne out by a recent result of Shapiro and Henneberger [15].

References

[1] Dirac P A M 1931 *Proc. R. Soc.* A **133** 60
[2] Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485
[3] Wu T T and Yang C N 1975 *Phys. Rev.* D **12** 3845
[4] Yang C N 1983 *Proc. Int. Symp. of Foundations of Quantum Mechanics (Tokyo 1983)* ed S Kamefuchi *et al* (Tokyo: Physical Society of Japan) p 5
[5] Henneberger W C 1981 *J. Math. Phys.* **22** 116
[6] Henneberger W C 1984 *Phys. Rev. Lett.* **52** 573
[7] Henneberger W C 1986 *Lett. Math. Phys.* **11** 309
[8] Wangsness R 1979 *Electromagnetic Fields* (New York: Wiley) p 327
[9] Boyer T H 1973 *Phys. Rev.* D **8** 1667
[10] Peshkin M 1981 *Phys. Rep.* **80** 377
[11] Freeman G 1986 private communication
[12] Coulson L and Freeman G 1986 private communication
[13] Jauch J M 1968 *Foundations of Quantum Mechanics* (Reading, MA: Addison-Wesley) p 185
[14] Peshkin M, Talmi I and Tassie L J 1961 *Ann. Phys., NY* **12** 426
[15] Shapiro D and Henneberger W C 1989 *J. Phys. A: Math. Gen.* **22** 3605